Is there a spin-glass phase in the random temperature Ising ferromagnet?

2002 J. Phys. A: Math. Gen. 35 1627

(http://iopscience.iop.org/0305-4470/35/7/311)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 1627–1649 PII: S0305-4470(02)28517-0

Is there a spin-glass phase in the random temperature Ising ferromagnet?

Gilles Tarjus and Victor Dotsenko[1]

Laboratoire de Physique Theorique des Liquides, UMR 7600, Universite Paris VI, 4 place Jussieu, 75252 Paris Cedex 05, France

Received 4 September 2001, in final form 5 December 2001
Published 8 February 2002
Online at stacks.iop.org/JPhysA/35/1627

Abstract

In this paper we study the phase diagram of the disordered Ising ferromagnet. Within the framework of the Gaussian variational approximation it is shown that in systems with a finite value of the disorder in dimensions $D = 4$ and $D < 4$ the paramagnetic and ferromagnetic phases are separated by a spin-glass phase. The transition from paramagnetic to spin-glass state is continuous (second order), whereas the transition between spin-glass and ferromagnetic states is discontinuous (first order). It is also shown that within the considered approximation there is no replica symmetry breaking in the spin-glass phase. The validity of the Gaussian variational approximation for the present problem is discussed, and we provide a tentative physical interpretation of the results.

PACS numbers: 75.10.Nr, 05.50.+q, 64.60−I, 75.50.−y

1. Introduction

Phase transitions in the random temperature Ising ferromagnets have been intensively studied theoretically, numerically and experimentally during the last decades. The theoretical interest has mainly been focused on the critical behaviour in the vicinity of the paramagnetic–ferromagnetic phase transition point T_c in *weakly* disordered systems [1]. Renormalization group considerations show that if the temperature is not too close to T_c, the critical behaviour is essentially controlled by the fixed point of the pure system (so that disorder produces only irrelevant corrections), while in the close vicinity of T_c the critical behaviour turns out to be different from that of the pure system and is characterized by a new universal (independent of the disorder strength) fixed point.

On these grounds it is widely believed that the critical behaviour of the disordered system is *universal*, and the strength of the disorder affects only the size of the critical region near T_c (but not the critical behaviour itself). In other words, the critical behaviour of systems with a finite value of the disorder must be the same as that of the weakly disordered systems. Most of

[1] On leave from: Landau Institute for Theoretical Physics, Moscow.

the numerical simulations (in particular for the two-dimensional systems) support this idea, see e.g. [2], although some of the numerical results seem to indicate that the critical behaviour can be non-universal and characterized by critical exponents depending on the disorder strength [3].

In this paper we consider the problem of the phase transitions in the disordered Ising ferromagnet from a somewhat different point of view. Instead of studying the critical behaviour, we propose first to address a simpler point concerning the nature of the phases in such systems. We stress that the usual assumption that the random temperature Ising ferromagnet can only be in the paramagnetic or in the ferromagnetic state is, at least, questionable, a point first raised by Ma and Rudnick [4]. Before studying such details as the critical properties, one should first clarify what kind of phases and what kind of transitions can exist in such a system.

A general reason for asking such a question comes from the fact that the saddle-point equations which describe the local minima of the disordered Hamiltonian in the (supposed) paramagnetic region have an exponentially large number of solutions. Physically this situation is quite clear: due to the spatial fluctuations of the local transition temperatures one can find a macroscopic number of 'ferromagnetic islands', well separated in space, that spend most of the time in a state with a non-zero local magnetization which can either be positive or negative. As long as these islands are rare (i.e. away from the supposed ferromagnetic transition temperature), they lead to the existence of an exponential number of local minima. Moreover, the presence of rare exponentially large islands results in the existence of non-analytic (Griffith-like) contributions to the thermodynamic functions [5].

An indirect indication that the phase behaviour of such systems could be more complicated than described by the renormalization group has been obtained in the framework of the so-called non-perturbative renormalization group (RG) approach [6]. In the latter, the existence of many different local minima of the disordered Hamiltonian is taken into account in the form of a replica symmetry breaking scheme, and it was eventually found that the renormalization flow leads to the strong coupling regime at the *finite* spatial scale, and not to the expected fixed points. This may indicate that something is basically wrong with the supposed (trivial) minimum of the renormalized Hamiltonian.

It has been suggested that on lowering the temperature, the localized ferromagnetic islands become close and strongly interacting, which leads to a transition to the global ferromagnetic state [7]. The solution of the saddle-point equations within the Gaussian variational approximation and the replica framework described in the next sections show that this is not the only possibility. We indeed find that upon lowering the temperature the global state of the system can become a spin glass *before* the ferromagnetic state sets in. In this spin-glass state the total magnetization remains zero, and there is an effective freezing of local (random) spin configurations (which leads to a non-zero value of the spin-glass Edwards–Anderson order parameter). Besides, one finds that in this spin-glass state the two-point spin–spin correlation function is described by a temperature-independent finite correlation length (it is interesting to note that this length coincides with that at which the strong coupling regime of the RG approach [6] sets in), whereas the (spin-glass type) four-spin correlation function becomes critical at the spin-glass phase transition point. Finally, on lowering the temperature further the global ferromagnetic state eventually sets in via a *first-order* phase transition.

The existence of an intermediate spin-glass phase in a system where *a priori* no frustrations, no competition of interactions occur is puzzling. This was stressed by Sherrington [8] in a response to the perturbative analysis (not within the framework of the replica method) of Ma and Rudnick [4] that predicted such a spin-glass phase. Besides the potential flaws associated with the perturbative treatments, the problem lies in the fact that it is hard to imagine a disordered ferromagnet in a state where the 4-spin spin-glass susceptibility is

larger than the square of the 2-spin ferromagnetic susceptibility [8] nor in a state with a zero total magnetization and a non-zero spin-glass order parameter. This will be discussed later, but, at the level of 'hand-waving arguments', one can propose a possible interpretation for the presence of a spin-glass phase. The point is that as one lowers the temperature (from the paramagnetic phase side) and the ferromagnetic islands become close and strongly interacting, there need not be the appearance of a unique infinite (percolating) ferromagnetic island. The existence of the spin-glass solution of the saddle-point equations in the considered disordered Ising ferromagnet requires that a large number of effectively independent spanning ferromagnetic clusters appear in the system. Just below the transition each of the clusters is characterized by non-zero value of its own global magnetization (so that within the cluster the spins are effectively 'frozen'), but the sign of these magnetizations remains random from cluster to cluster. This situation manifests itself as the spin-glass state with 'frozen' spins and no (averaged-over clusters) global magnetization. Finally, when the temperature is further decreased, the effective interactions among these spanning clusters become strong enough for the system to eventually make a 'jump' (via a first-order transition) into the ferromagnetic state. In other words, the ferromagnetic phase sets in due to a collective locking of the orientations of the clusters magnetizations in the same direction. It is easy to understand that this transition must be first order. Indeed, since at the point of the spin-glass to ferromagnetic transition the absolute value of the (randomly directed) magnetizations of the spanning ferromagnetic clusters in the spin-glass phase is already finite, the value of the global ferromagnetic order parameter resulting from the locking of the various orientations in the same direction is itself finite.

In the next section we present the general formalism in terms of the standard replica approach and of the Gaussian variational approximation [9, 10] as applied to the random temperature model. In section 3 we derive all the solutions of the corresponding saddle-point equations, solutions that describe the different 'ground states' that can exist in the model. It is shown that the spin-glass solution discussed above can exist only in dimensions $D \leqslant 4$. Since the conclusions of the present study are, to a large extent, only of a qualitative nature, we focus on the system in dimension $D = 4$ (the generalization of the results for dimensions $D = (4 - \epsilon)$ is given in appendix C). We obtain the solutions for the paramagnetic, (replica-symmetric) spin-glass and ferromagnetic states, and we derive the temperature regions over which these phases are stable as well as the nature of the phase transitions separating these phases. In section 4 the singularity in the spin-glass-type 4-spin correlation function and in the corresponding susceptibility at the spin-glass phase transition is derived. (In appendix A we give the formal proof that, in the framework of the present formalism, no replica symmetry breaking solutions, either continuous or step-like, can exist in the spin-glass state.) Finally, in section 5 we discuss the validity of the Gaussian variational method; we stress, in particular, that the present approach can only be reliable for finite values of the parameter describing the disorder strength. We also suggest a possible scenario for the existence of an intermediate spin-glass phase.

2. General formalism

In this paper we study the disordered (random temperature) D-dimensional Ising ferromagnet which can be described in the continuum by the following Ginsburg–Landau Hamiltonian:

$$H[\phi(x); \delta\tau(x)] = \int \mathrm{d}^D x \left[\tfrac{1}{2}(\nabla\phi(x))^2 + \tfrac{1}{2}(\tau - \delta\tau(x))\phi^2(x) + \tfrac{1}{4}g\phi^4(x)\right]. \tag{2.1}$$

Here, $\tau \equiv (T - T_c)/T_c \ll 1$ is the reduced temperature, and the quenched disorder is described by random spatial fluctuations of the local transition temperature $\delta\tau(x)$ whose probability distribution is taken to be symmetric and Gaussian:

$$P[\delta\tau] = p_0 \exp\left(-\frac{1}{4u}\int \mathrm{d}^D x (\delta\tau(x))^2\right) \tag{2.2}$$

where u is the parameter which describes the strength of the disorder and p_0 is an irrelevant normalization constant.

In terms of the standard replica method, the averaged (over quenched disorder) free energy is calculated from an annealed average involving n copies of the same system:

$$F = -\overline{(\ln Z)} = -\lim_{n\to 0}\frac{1}{n}\ln[\overline{Z^n}] \tag{2.3}$$

where $\overline{(\ldots)}$ denotes the averaging over the random function $\delta\tau(x)$ with the probability distribution (2.2), and

$$\overline{Z^n} \equiv \int \mathcal{D}\delta\tau(x) P[\delta\tau]\left[\int \mathcal{D}\phi(x)\exp(-H[\phi(x);\tau(x)])\right]^n \tag{2.4}$$

is the replica partition function. Simple Gaussian integration over $\delta\tau(x)$ in equation (2.4) yields

$$\overline{Z^n} = \prod_{a=1}^{n}\left[\int \mathcal{D}\phi_a(x)\right]\exp\bigl(-H^{(n)}[\phi_a(x)]\bigr) \tag{2.5}$$

where

$$H^{(n)}[\phi_a(x)] = \int \mathrm{d}^D x\left[\frac{1}{2}\sum_{a=1}^{n}(\nabla\phi_a)^2 + \frac{1}{2}\tau\sum_{a=1}^{n}\phi_a^2 + \frac{1}{4}\sum_{a,b=1}^{n} g_{ab}\phi_a^2\phi_b^2\right] \tag{2.6}$$

is the replica Hamiltonian and

$$g_{ab} = g\delta_{ab} - u. \tag{2.7}$$

To take into account the possibility of ferromagnetic ordering in the system we explicitly introduce the ferromagnetic order parameter $m = \overline{\langle\phi\rangle}$ by redefining the fields as follows:

$$\phi_a(x) = m + \varphi_a(x) \tag{2.8}$$

where the new fields $\varphi_a(x)$ describe the spatial fluctuations with zero mean. By substituting (2.8) into equation (2.6) for the replica Hamiltonian, one finds

$$\begin{aligned} H^{(n)}[\varphi_a(x);m] = Vn\left(\frac{1}{2}\tau m^2 + \frac{1}{4}gm^4\right) + \int \mathrm{d}^D x\Bigg[&\frac{1}{2}\sum_{a=1}^{n}(\nabla\varphi_a)^2 + \frac{1}{2}\tau\sum_{a=1}^{n}\varphi_a^2 \\ &+ \frac{1}{4}\sum_{a,b=1}^{n} g_{ab}\varphi_a^2\varphi_b^2 + \frac{1}{2}gm^2\sum_{a=1}^{n}\varphi_a^2 + m^2\sum_{a,b=1}^{n} g_{ab}\varphi_a\varphi_b + \tau m\sum_{a=1}^{n}\varphi_a \\ &+ m\sum_{a,b=1}^{n} g_{ab}\varphi_a\varphi_b^2 + m^3 g\sum_{a=1}^{n}\varphi_a\Bigg] \end{aligned} \tag{2.9}$$

where V is the volume of the system. Note that the limit $n \to 0$, that must formally be taken in the final results, allows us to omit all terms of order n^2 in the above expression (and in further calculations).

The idea of the Gaussian variational approach is to approximate the fluctuations of the fields $\varphi_a(x)$ in the above equation (2.9) by the Gaussian trial Hamiltonian

$$H_g^{(n)}[\varphi_a|\mathbf{G}] = \frac{V}{2}\int_{|p|<1}\frac{\mathrm{d}^D p}{(2\pi)^D}\sum_{a,b=1}^{n} G_{ab}^{-1}(p)\varphi_a(p)\varphi_b(-p) \tag{2.10}$$

where the correlation functions $G_{ab}(p) = \langle\varphi_a(p)\varphi_b(-p)\rangle$ are considered as variational parameters.

The replica partition function can be represented as follows:

$$\overline{Z^n} = \prod_{a=1}^{n}\left[\int \mathcal{D}\varphi_a(x)\right]\exp\left\{-H_g^{(n)}[\varphi_a(x)] - \left(H^{(n)}[\varphi_a(x);m] - H_g^{(n)}[\varphi_a(x)]\right)\right\} \tag{2.11}$$

and in the first-order cumulant approximation in the difference $\left(H^{(n)} - H_g^{(n)}\right)$ one finds

$$\overline{Z^n} \simeq \exp\left[-\frac{1}{2}V\int_{|p|<1}\frac{\mathrm{d}^D p}{(2\pi)^D}\mathrm{Tr}\,\ln(\mathbf{G}^{-1}(p)) - \left\langle\left(H^{(n)} - H_g^{(n)}\right)\right\rangle_g\right] \equiv \exp(-nVf[m;\mathbf{G}]) \tag{2.12}$$

where $\langle(\ldots)\rangle_g$ denotes the averaging with the Gaussian weight, equation (2.10); $f[m;\mathbf{G}]$ is the density of free energy that depends on the order parameter m and on the trial correlation functions $G_{ab}(p)$:

$$f[m;\mathbf{G}] = \frac{1}{2n}\int_{|p|<1}\frac{\mathrm{d}^D p}{(2\pi)^D}\mathrm{Tr}\,\ln(\mathbf{G}^{-1}(p)) + \frac{1}{nV}\left\langle\left(H^{(n)} - H_g^{(n)}\right)\right\rangle_g. \tag{2.13}$$

Since the above free energy density is an upper bound of the exact replica free energy density, the variational parameters m and $G_{ab}(p)$ can be determined by minimization of equation (2.13). One should however keep in mind the oddities related to the limit $n \to 0$, in particular the fact that the number of parameters can turn negative for $n < 1$ (see section 3.3)[2]. Inserting equations (2.9) and (2.10) into equation (2.13) leads to

$$\begin{aligned} f[m;\mathbf{G}] = {} & -\frac{1}{2n}\int_{|p|<1}\frac{\mathrm{d}^D p}{(2\pi)^D}\mathrm{Tr}\,\ln(\mathbf{G}(p)) + \frac{1}{2}\tau m^2 + \frac{1}{4}gm^4 \\ & + \frac{1}{2n}\int_{|p|<1}\frac{\mathrm{d}^D p}{(2\pi)^D}(p^2+\tau)\sum_{a=1}^{n}\langle\varphi_a(p)\varphi_a(-p)\rangle_g + \frac{1}{4n}\sum_{a,b=1}^{n} g_{ab}\left\langle\varphi_a^2(x)\varphi_b^2(x)\right\rangle_g \\ & + \frac{1}{2n}gm^2\sum_{a=1}^{n}\left\langle\varphi_a^2(x)\right\rangle_g + \frac{1}{n}m^2\sum_{a,b=1}^{n} g_{ab}\langle\varphi_a(x)\varphi_b(x)\rangle_g. \end{aligned} \tag{2.14}$$

Above and in what follows we omit irrelevant constant terms. For the Gaussian averages of the fluctuating fields, one has

$$\langle\varphi_a(x)\varphi_b(x)\rangle_g = \int_{|p|<1}\frac{\mathrm{d}^D p}{(2\pi)^D}G_{ab}(p) \equiv [G_{ab}] \tag{2.15}$$

$$\left\langle\varphi_a^2(x)\right\rangle_g = \int_{|p|<1}\frac{\mathrm{d}^D p}{(2\pi)^D}G_{aa}(p) \equiv [G_{aa}] \tag{2.16}$$

$$\left\langle\varphi_a^2(x)\varphi_b^2(x)\right\rangle_g = \left\langle\varphi_a^2(x)\right\rangle_g\left\langle\varphi_b^2(x)\right\rangle_g + 2\langle\varphi_a(x)\varphi_b(x)\rangle_g^2 \equiv [G_{aa}][G_{bb}] + 2[G_{ab}]^2 \tag{2.17}$$

[2] As noted by Mezard and Parisi in their replica field theory for random manifolds [10], the Gaussian variational method becomes exact when the number N of components of the fields ϕ_a goes to infinity (here, $N = 1$). To apply this remark to the present case, one must generalize the non-Gaussian term appearing in the replica Hamiltonian, equation (2.6), to $\frac{1}{12}\sum_{ab=1}^{n}(g_{ab}/N)\left[\phi_a^2\phi_b^2 + 2(\phi_a\cdot\phi_b)^2\right]$. Note that with this latter term the Hamiltonian *does not* correspond to the replica-space formulation of the random temperature $O(N)$ model.

where we have introduced the notation

$$\int_{|p|<1} \frac{\mathrm{d}^D p}{(2\pi)^D} A(p) \equiv [A] \tag{2.18}$$

for an arbitrary function $A(p)$. Taking into account that the diagonal elements of the matrix $\mathbf{G}$ must be independent of the replica index, $G_{aa} \equiv \tilde{G}$, we find the following expression for the free energy density:

$$f[m;\mathbf{G}] = -\frac{1}{2n}\mathrm{Tr}\,[\ln(\mathbf{G})] + \frac{1}{2}\tau m^2 + \frac{1}{4}gm^4 + \frac{1}{2}[(p^2+\tau)\tilde{G}] + \frac{1}{4}g[\tilde{G}]^2 + \frac{1}{2}(g-u)[\tilde{G}]^2 + \frac{1}{2n}\sum_{a\neq b}^{n} g_{ab}[G_{ab}]^2 + \frac{1}{2}gm^2[\tilde{G}] + m^2(g-u)[\tilde{G}] + \frac{1}{n}m^2\sum_{a\neq b}^{n} g_{ab}[G_{ab}]. \tag{2.19}$$

The correlation functions $G_{ab}(p)$ and the order parameter m are then determined by the following saddle-point equations:

$$\frac{\delta f}{\delta \tilde{G}(p)} = 0 \tag{2.20}$$

$$\frac{\delta f}{\delta G_{ab}(p)} = 0 \qquad (a \neq b) \tag{2.21}$$

$$\frac{\delta f}{\delta m} = 0. \tag{2.22}$$

By using the explicit expression of the free energy density, equation (2.19), one obtains

$$G_{ab}^{-1}(p) = (p^2+\tau)\delta_{ab} + g[\tilde{G}]\delta_{ab} + 2g_{ab}[G_{ab}] + gm^2\delta_{ab} + 2m^2 g_{ab} \tag{2.23}$$

$$m\left(\tau + gm^2 + (3g-2u)[\tilde{G}] + \frac{2}{n}\sum_{a\neq b}^{n} g_{ab}[G_{ab}]\right) = 0. \tag{2.24}$$

According to equation (2.23) one finds that the trial correlation function has the following structure:

$$G_{ab}^{-1}(p) = (p^2+\tau)\delta_{ab} + \mu_{ab} \tag{2.25}$$

where the matrix μ_{ab} is defined by

$$\mu_{ab} = (g[\tilde{G}] + gm^2)\delta_{ab} + 2g_{ab}[G_{ab}] + 2g_{ab}m^2. \tag{2.26}$$

For finding explicit solutions of this equation one needs to make an assumption about the replica structure of the matrix μ_{ab}. In what follows we assume that this matrix is replica symmetric; in appendix A we give the formal proof that equation (2.26) has no solutions with the Parisi replica symmetry breaking structure for the matrix μ_{ab}. The replica symmetric ansatz implies that the matrix μ_{ab} is defined by only two parameters,

$$\mu_{ab} = (\tilde{\mu}+\mu)\delta_{ab} - \mu = \begin{cases} \tilde{\mu} & a = b \\ -\mu & a \neq b. \end{cases} \tag{2.27}$$

For the corresponding replica symmetric correlation function, defined by equation (2.25), we find

$$\begin{aligned} G_{ab}(p;\lambda,\mu) &= \frac{1}{p^2+\lambda}\delta_{ab} + \frac{\mu}{(p^2+\lambda)^2} \\ &\equiv G_c(p;\lambda)\delta_{ab} + \mu(G_c(p;\lambda))^2 \end{aligned} \tag{2.28}$$

where the so-called 'connected' part of the correlation function is given by

$$G_c(p;\lambda) = \frac{1}{p^2+\lambda} \tag{2.29}$$

and instead of $\tilde{\mu}$ (defined in equation (2.27)) we have introduced a physically motivated 'mass' parameter $\lambda = \tau + \tilde{\mu} + \mu$ that determines the value of the correlation length ($R_c \sim \lambda^{-1/2}$ in the present approximation).

Note that according to equation (2.28) the parameter μ is related to the value of the spin-glass Edwards–Anderson (EA) order parameter, since

$$q = \overline{\langle\varphi\rangle^2} = \lim_{n\to 0}\langle\varphi_a(x)\varphi_b(x)\rangle|_{(a\neq b)} = \mu\left[G_c^2\right]. \tag{2.30}$$

Thus, to be physically meaningful μ must be non-negative.

By using equations (2.27), (2.28) and (2.7), the corresponding saddle-point equations for the parameters m, λ and μ can be obtained from equations (2.26) and (2.24) as

$$\lambda = \tau + \mu + (3g - 2u)\left([G_c] + \mu\left[G_c^2\right]\right) + (3g - 2u)m^2 \tag{2.31}$$

$$\mu = 2u\mu\left[G_c^2\right] + 2um^2 \tag{2.32}$$

$$m\left(\tau + gm^2 + (3g - 2u)\left([G_c] + \mu\left[G_c^2\right]\right) + 2u\mu\left[G_c^2\right]\right) = 0. \tag{2.33}$$

The resulting free energy density is then given by

$$\begin{aligned} f(m,\lambda,\mu) = {} & -\tfrac{1}{2}[\ln(G_c)] + \tfrac{1}{2}(\tau - \lambda)\left([G_c] + \mu\left[G_c^2\right]\right) + \tfrac{1}{4}(3g - 2u)\left([G_c] + \mu\left[G_c^2\right]\right)^2 \\ & + \tfrac{1}{2}u\mu^2\left[G_c^2\right] + \tfrac{1}{2}(3g - 2u)m^2\left([G_c] + \mu\left[G_c^2\right]\right) + u\mu m^2\left[G_c^2\right] \\ & + \tfrac{1}{2}\tau m^2 + \tfrac{1}{4}gm^4. \end{aligned} \tag{2.34}$$

3. Phase diagram in $D = 4$

In this section we study all possible solutions of the saddle-point equations (2.31)–(2.33). As usual, it is assumed that the non-Gaussian coupling parameters g and u of the original replica Hamiltonian, equation (2.6), are small: $g \ll 1, u \ll 1$. Besides, we consider $u \sim g$, which qualitatively corresponds to the situation of 'finite disorder strength' since then the parameter u describing the disorder strength is of the same order as the coupling parameter g of the pure system. As will be discussed in section 5, the analysis of the validity of the present (first-order) Gaussian approximation shows that it may give reasonable results provided the ratio u/g stays within certain numerical bounds (see section 5). For the moment, however, it is sufficient to assume that $u < \frac{3}{2}g$ (see below). Finally, since we are only interested in the large-scale (continuous limit) properties of the system we consider the region of parameter space where the mass λ of the connected correlation function is also small: $\lambda \ll 1$.

To simplify the algebra and for a qualitative presentation of the phase diagram, it is convenient to consider first the solutions of the saddle-point equations in dimension $D = 4$. Generalization of the results for dimensions below four will be given in section 5 (it will also be shown that for $D > 4$ the spin-glass solution does not exist).

For $D = 4$ and for $\lambda \ll 1$ one has

$$[G_c] = \int_{|p|<1}\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{p^2+\lambda} \simeq C\left(1 - \lambda\ln\left(\frac{1}{\lambda}\right)\right) \tag{3.1}$$

$$\left[G_c^2\right] = \int_{|p|<1}\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{(p^2+\lambda)^2} \simeq C\left(\ln\left(\frac{1}{\lambda}\right) - 1\right) \tag{3.2}$$

where $C = 1/16\pi^2$.

3.1. Paramagnetic solution

In the paramagnetic state, the ferromagnetic and spin-glass order parameters are both zero ($m = \mu = 0$) and there is only one saddle-point equation (2.31) for the mass parameter λ,

$$\lambda = \tau + (3g - 2u)[G_c]. \tag{3.3}$$

Using equation (3.1) leads to the following equation:

$$\lambda + C(3g - 2u)\lambda \ln\left(\frac{1}{\lambda}\right) = \tau + C(3g - 2u) \tag{3.4}$$

which provides the dependence $\lambda = \lambda(\tau)$. The solution of this equation makes physical sense only for $\lambda \geqslant 0$, and therefore this condition defines the limit of existence of the paramagnetic phase. Provided $3g > 2u$ the above equation yields positive (physical) solutions for $\lambda(\tau)$ only for temperatures such that

$$\tau \geqslant \tau_c = -C(3g - 2u). \tag{3.5}$$

If one finds that the ferromagnetic solution appears just below τ_c, then the temperature $\tau = \tau_c$ would correspond to the paramagnetic–ferromagnetic phase transition point. However, it will be shown below that this is not the case. In fact, below a certain temperature $\tau_{sg} > \tau_c$ a spin-glass solution (with $\mu \neq 0$) appears, and at temperatures $\tau < \tau_{sg}$ it is the spin-glass state that turns out to be stable, while the paramagnetic state becomes unstable.

3.2. Spin-glass solution

The spin-glass state is defined by two saddle-point equations (2.31) and (2.32):

$$\lambda = \tau + \mu + (3g - 2u)\left([G_c] + \mu\left[G_c^2\right]\right) \tag{3.6}$$

$$\mu = 2u\mu\left[G_c^2\right] \tag{3.7}$$

which define two non-zero (positive) order parameters: $\mu(\tau)$ and $\lambda(\tau)$. From the last equation one immediately finds that for $\mu \neq 0$ the mass parameter λ becomes temperature independent, $\lambda = \lambda_o$, and the value of λ_o is defined by the condition

$$\left[G_c^2\right] = \frac{1}{2u}. \tag{3.8}$$

Correspondingly, equation (3.6) yields the following solution for the spin-glass order parameter:

$$\mu(\tau) = \frac{2u}{3g}(\lambda_o - (3g - 2u)[G_c] - \tau). \tag{3.9}$$

By making use of equations (3.1) and (3.2) one can find the solutions for λ_o and $\mu(\tau)$ explicitly:

$$\lambda_o = \exp\left(-\frac{1}{2Cu} - 1\right) \tag{3.10}$$

$$\mu(\tau) = \frac{2u}{3g}(\tau_{sg} - \tau) \tag{3.11}$$

where

$$\tau_{sg} = \lambda_o - (3g - 2u)[G_c] = \tau_c + \left(\frac{3g}{2u} + C(3g - 2u)\right)\lambda_o > \tau_c \tag{3.12}$$

and τ_c is the putative paramagnetic critical point discussed above. The solution for $\mu > 0$ appears (i.e. becomes physical) only for $\tau < \tau_{sg}$, and therefore the point τ_{sg} can be associated

with the spin-glass phase transition temperature. Note that according to equations (3.12) and (3.3), the value of λ in the paramagnetic phase at $\tau = \tau_{sg}$ is equal to λ_o (for $\tau > \tau_{sg}$, $\lambda(\tau) > \lambda_o$ and for $\tau < \tau_{sg}$, $\lambda(\tau) < \lambda_o$). Since $\mu(\tau_{sg}) = 0$ whether one comes from the paramagnetic or the spin-glass phase, the transition into the spin-glass phase is clearly continuous.

In appendix B we present a detailed study of the stability of the spin-glass and the paramagnetic solutions obtained above. It is shown there that for $\tau > \tau_{sg}$ the only stable state of the system is paramagnetic, while for $\tau < \tau_{sg}$ the paramagnetic solution becomes unstable and the stable state of the system is the spin-glass phase.

3.3. Ferromagnetic solution

We finally consider the ferromagnetic solution of the saddle-point equations (2.31)–(2.33) in which all three parameters λ, μ and m are non-zero. After some simple algebra we find

$$m^2 = \frac{1}{2g}\lambda \tag{3.13}$$

$$\mu = \frac{u}{g}\frac{\lambda}{1 - 2u[G^2]} \tag{3.14}$$

where the parameter $\lambda(\tau)$ is obtained from the following equation:

$$\lambda = \tau + (3g - 2u)[G] + \frac{3}{2}\frac{\lambda}{1 - 2u[G^2]}. \tag{3.15}$$

Substituting equations (3.1) and (3.2) into the above equation gives

$$\lambda\left[\frac{3g}{2u} + C(3g - 2u) - C(3g - 2u)\ln\frac{\lambda}{\lambda_o} - \frac{3}{4Cu}\frac{1}{\ln\frac{\lambda}{\lambda_o}}\right] = \tau + C(3g - 2u). \tag{3.16}$$

A simple analysis shows that upon lowering the temperature τ a solution of this equation appears for the first time below a temperature τ_*. This solution has a *finite* (non-zero) value λ_* at $\tau = \tau_*$, which indicates that the phase transition into the ferromagnetic state is first order. To leading order in $g \ll 1$ and in $u \ll 1$ (and for $g/u \sim 1$) one finds

$$\tau_* \simeq -C(3g - 2u) - \frac{3}{4Cu}\exp\left(-\frac{1}{2Cu}\right) = \tau_c - \frac{3\mathrm{e}}{4Cu}\lambda_o < \tau_c \tag{3.17}$$

and

$$\lambda(\tau = \tau_*) \equiv \lambda_* \simeq \exp\left(-\frac{1}{2Cu}\right) = \mathrm{e}\lambda_o. \tag{3.18}$$

By inserting the above value into equations (3.13) and (3.14), we find the corresponding values of the ferromagnetic and the spin-glass order parameters:

$$m_*^2 = \frac{1}{2g}\lambda_* \simeq \frac{1}{2g}\exp\left(-\frac{1}{2Cu}\right) \tag{3.19}$$

$$\mu_* \simeq \frac{\mathrm{e}}{2Cg}\lambda_o = \frac{1}{2Cg}\exp\left(-\frac{1}{2Cu}\right). \tag{3.20}$$

Straightforward calculations similar to those of section 3.3 show that the ferromagnetic solution defined by equations (3.13)–(3.15) is stable at all temperatures $\tau < \tau_*$. Thus, below τ_* both the spin-glass and the ferromagnetic solutions are (locally) stable (this is the standard situation for first-order phase transitions). To determine which of these two states is the global minimum of the free energy at a given temperature, we have to compare the corresponding values of their free energies.

3.4. First-order phase transition between spin-glass and ferromagnetic states

Substituting the spin-glass solution, equations (3.8)–(3.12), into equation (2.34) provides the value of the free energy density of the spin-glass state (in the leading order in $g, u \ll 1$):

$$f_{sg}(t) \simeq f_0(\tau) + \frac{1}{8}\lambda_o^2 - \frac{Cu}{3g}\lambda_o(\tau - \tau_c) \tag{3.21}$$

where

$$f_0(\tau) = \frac{u}{6g}C(3g - 2u) + \frac{u}{3g}\tau - \frac{1}{12}\tau^2. \tag{3.22}$$

On the other hand, for the ferromagnetic solution, equations (3.13)–(3.16), one gets

$$f_f(\tau) \simeq f_0(\tau) + \frac{\lambda}{6g}(1 + 2Cu\ln\lambda)(\tau - \tau_c) + \frac{C^2u(3g - 2u)}{6g}\lambda^2\ln^2\lambda + C\left(\frac{(3g - 2u}{6g} - \frac{1}{4}\right)\lambda^2\ln\lambda + \frac{1}{24g}\lambda^2 - \frac{1}{8}\lambda^2 \tag{3.23}$$

where the value of $\lambda(\tau)$ is given by equation (3.16). Let us redefine

$$\lambda(\tau) \equiv \lambda_o x(\tau) \tag{3.24}$$

$$\tau - \tau_c \equiv -\lambda_o t \tag{3.25}$$

By making the above change of variables in the saddle-point equation (3.16), one obtains

$$x\left[\frac{3g}{2u} + C(3g - 2u)(1 - \ln x) - \frac{3}{4Cu}\frac{1}{\ln x}\right] = t. \tag{3.26}$$

Therefore, to leading order in $u, g \ll 1$ the value of the parameter x as a function of the reduced temperature t is defined by the following equation:

$$\frac{3}{4Cu}\frac{x}{\ln x} \simeq t. \tag{3.27}$$

Assuming that at the point of the phase transition, i.e., when $f_f = f_{sg}$, the value of the parameter $x(t)$ is of the order one, we find for the difference of the free energies, equations (3.21) and (3.23) (in the leading order in $u, g \ll 1$ and $\lambda_o \ll 1$),

$$f_f - f_{sg} \simeq \frac{Cu}{3g}\lambda_o^2 t\,(1 - x - x\ln x) + \frac{1}{8g}\lambda_o^2 x^2. \tag{3.28}$$

Thus, the transition point ($f_f - f_{sg} = 0$) is defined by the following equation:

$$\frac{C}{3}ut\,(1 - x - x\ln x) = \frac{1}{8}x^2. \tag{3.29}$$

Combining equations (3.27) and (3.29), we finally derive the equation for the parameter x at the phase transition point,

$$\frac{3}{2}x - x\ln x = 1. \tag{3.30}$$

This equation has a unique solution $x = x_f \sim 1$ ($x_f > 1$). Substituting x_f into equations (3.27) and (3.25) one obtains the temperature of the (first-order) phase transition between the spin-glass and the ferromagnetic phases:

$$\tau_f = \tau_c - \frac{3}{4Cu}\frac{x_f}{\ln x_f}\lambda_o \tag{3.31}$$

which is less than τ_*.

4. Singularities at the spin-glass phase transition

Within the Gaussian variational approximation the (connected) correlation functions can be obtained by adding source terms to the replica Hamiltonian, equation (2.6), and by approximating the free-energy functional to the first-order cumulant as in equation (2.12). This latter then generates the (connected) correlation functions that are obtained by functional differentiation with respect to the source terms. By introducing source terms linearly coupled to the fields $\phi_a(x)$, one derives the usual correlation functions, whose expression (for the two-point functions) coincides with that given in section 2. To study the singularity at the spin-glass transition it is more convenient to introduce source terms linearly coupled to the composite operators $\frac{1}{2}\phi_a(x)\phi_b(x)$. This leads us to the following replica Hamiltonian:

$$H^{(n)}[\phi_a(x);\boldsymbol{\Delta}(x)] = H^{(n)}[\phi_a(x);\mathbf{0}] - \frac{1}{2}\int \mathrm{d}^D x \sum_{a,b=1}^{n} \Delta_{ab}(x)\phi_a(x)\phi_b(x) \tag{4.1}$$

where $H^{(n)}[\phi_a(x);\mathbf{0}]$ is given by equation (2.6) and, because we are ultimately interested in the replica-symmetric solution, the source term is taken as

$$\Delta_{ab}(x) = (\tilde{\Delta}(x) + \Delta(x))\delta_{ab} - \Delta(x). \tag{4.2}$$

Since we study here only the paramagnetic and the spin-glass phases, we can set $m = 0$. A Gaussian trial Hamiltonian is chosen as before, but the presence of space-dependent source terms breaks the translational invariance and requires trial Green functions that depend on two space points:

$$H_g^{(n)}[\phi_a|\mathbf{G}] = \frac{1}{2}\int\int \mathrm{d}^D x\, \mathrm{d}^D x' \sum_{ab=1}^{n} \phi_a(x)[\mathbf{G}^{-1}]_{ab}^{xx'}\phi_b(x') \tag{4.3}$$

where

$$[\mathbf{G}^{-1}]_{ab}^{xx'} = \left[\left(\frac{\partial^2}{\partial x\,\partial x'} + \tau\right)\delta_{ab} + \mu_{ab}(x)\right]\delta(x - x') \tag{4.4}$$

which simply generalizes equation (2.25). For the replica-symmetric solution, the variational parameter $\mu_{ab}(x)$ can be written as

$$\mu_{ab}(x) = (\tilde{\mu}(x) + \mu(x))\delta_{ab} - \mu(x) \tag{4.5}$$

whereas the Green functions can be written as

$$G_{ab}^{xx'} = G_c^{xx'}\delta_{ab} + G_d^{xx'} \tag{4.6}$$

where G_c and G_d represent, as usual in the presence of quenched disorder, the 'connected' and 'disconnected' parts respectively and can be expressed in terms of τ, $\tilde{\mu}(x)$ and $\mu(x)$ by inverting equation (4.4).

One can then follow the procedure used in section 2: the first-order cumulant approximation in the deviation $\left(H^{(n)} - H_g^{(n)}\right)$ provides an upper bound for the free-energy functional, and minimizing with respect to the trial Green function elements leads to saddle-point equations that in the limit $n \to 0$ for a replica-symmetric scheme reduce to

$$\tilde{\mu}(x) + \tilde{\Delta}(x) = (3g - 2u)\left(G_c^{xx} + G_d^{xx}\right) \tag{4.7}$$

$$\mu(x) + \Delta(x) = 2uG_d^{xx}. \tag{4.8}$$

When considered at the saddle-point characterized by the above equations, the variational free energy $F[\tilde{\Delta}(x), \Delta(x)]$ can be used as the generating functional for the correlation

functions of the composite operators $\frac{1}{2}\phi_a(x)\phi_b(x)$. More precisely, one has

$$\frac{\delta F}{\delta \tilde{\Delta}(x)} = -\frac{1}{2}\lim_{n\to 0}\langle \phi_a(x)\phi_a(x)\rangle = -\frac{1}{2}\left(G_c^{xx} + G_d^{xx}\right) \tag{4.9}$$

$$\frac{\delta F}{\delta \Delta(x)} = \frac{1}{2}\lim_{n\to 0}\langle \phi_a(x)\phi_b(x)\rangle\Big|_{(a\neq b)} = \frac{1}{2}G_d^{xx}. \tag{4.10}$$

In the limit $\tilde{\Delta} = \Delta = 0$ the above equations reduce to the expressions already given in section 2, in particular equation (4.10) reduces to equation (2.30) for the spin-glass order parameter: indeed, in the absence of source terms translational invariance is recovered and $G_d^{xx} = [G_d(p)] = \mu\big[G_c^2(p)\big]$.

The second functional derivatives of F provide the wanted two-point, four-field correlation functions, namely

$$\frac{\delta^2 F}{\delta\tilde{\Delta}(x)\delta\tilde{\Delta}(x')} = -\frac{1}{4}\lim_{n\to 0}\frac{1}{n}\sum_{a,b=1}^{n}\left\langle \phi_a^2(x)\phi_b^2(x')\right\rangle_c \tag{4.11}$$

$$\frac{\delta^2 F}{\delta\tilde{\Delta}(x)\delta\Delta(x')} = \frac{1}{4}\lim_{n\to 0}\frac{1}{n}\sum_{a,b=1}^{n}\left\langle \phi_a^2(x)\left(\sum_{c\neq b}^{n}\phi_c(x')\phi_b(x')\right)\right\rangle_c \tag{4.12}$$

$$\frac{\delta^2 F}{\delta\Delta(x)\delta\Delta(x')} = -\frac{1}{4}\lim_{n\to 0}\frac{1}{n}\sum_{a,b=1}^{n}\left\langle \left(\sum_{c\neq a}^{n}\phi_a(x)\phi_c(x)\right)\left(\sum_{d\neq b}^{n}\phi_b(x')\phi_d(x')\right)\right\rangle_c \tag{4.13}$$

where $\langle(\ldots)\rangle_c$ denotes a cumulant average for the composite operators.

We are interested in the case $\tilde{\Delta} = \Delta = 0$, in which translational invariance is recovered and the two-point functions are diagonal in momentum space. We then define the 2×2 momentum-dependent susceptibility matrix $\boldsymbol{\chi}(p)$ by

$$\begin{aligned}
\chi_{11}(p) &= \frac{\delta^2 F}{\delta\tilde{\Delta}(-p)\delta\tilde{\Delta}(p)}\bigg|_{\tilde{\Delta}=\Delta=0} & \chi_{12}(p) &= \frac{\delta^2 F}{\delta\tilde{\Delta}(-p)\delta\Delta(p)}\bigg|_{\tilde{\Delta}=\Delta=0}\\
\chi_{21}(p) &= \frac{\delta^2 F}{\delta\Delta(-p)\delta\tilde{\Delta}(p)}\bigg|_{\tilde{\Delta}=\Delta=0} & \chi_{22}(p) &= \frac{\delta^2 F}{\delta\Delta(-p)\delta\Delta(p)}\bigg|_{\tilde{\Delta}=\Delta=0.}
\end{aligned} \tag{4.14}$$

The expression of these p-dependent susceptibilities in terms of the correlation functions follows from equations (4.11)–(4.13). By combining the above definitions, the expression of the free-energy functional at the first-order cumulant approximation and the saddle-point equations, equations (4.7)–(4.8), one obtains after some algebra the following expressions for the p-dependent susceptibility matrix:

$$\boldsymbol{\chi}(p) = \frac{1}{2}\begin{pmatrix} \frac{1}{(3g-2u)} & 0 \\ 0 & \frac{1}{2u}\end{pmatrix}\left(\mathbf{M}^{-1}(p) - \mathbf{I}\right) \tag{4.15}$$

where $\mathbf{I}$ is the identity matrix and the p-dependent matrix $\mathbf{M}(p)$ is defined by

$$\mathbf{M}(p) = \begin{pmatrix} 1 + (3g-2u)(I(p) + 2\mu J(p)) & 2(3g-2u)\mu J(p) \\ 4u\mu J(p) & 1 - 2u(I(p) - 2\mu J(p))\end{pmatrix} \tag{4.16}$$

with

$$I(p) = \int_{|p|<1}\frac{\mathrm{d}^D k}{(2\pi)^D}\frac{1}{(\mathbf{k}^2+\lambda)((\mathbf{k}+\mathbf{p})^2+\lambda)} \tag{4.17}$$

$$J(p) = -\frac{1}{2}\frac{\partial I(p)}{\partial\lambda} = \int_{|p|<1}\frac{\mathrm{d}^D k}{(2\pi)^D}\frac{1}{(\mathbf{k}^2+\lambda)^2((\mathbf{k}+\mathbf{p})^2+\lambda)} \tag{4.18}$$

and λ and μ are given by equations (3.6) and (3.7), respectively. From the above equations one immediately obtains the expression for the susceptibility matrix in the paramagnetic phase (where $\mu = 0$):

$$\chi(p) = \frac{1}{2}\begin{pmatrix} -\frac{I(p)}{1+(3g-2u)I(p)} & 0 \\ 0 & \frac{I(p)}{1-2uI(p)} \end{pmatrix} \tag{4.19}$$

from which one derives that the susceptibility $\chi_{22}(p=0) = (\partial^2 F/\partial\Delta^2)|_{(\tilde{\Delta}=\Delta=0)}$ diverges when $1 - 2uI(0) = 0$, i.e., by using equation (4.17), when $\left[G_c^2\right] = 1/2u$. This point is precisely attained at the transition to the spin-glass state, $\tau = \tau_{sg}$, where, for $D = 4$, τ_{sg} is given by equation (3.12) (λ is then equal to λ_o given by equation (3.10)). To derive the critical behaviour of $\chi_{22}(p)$ when approaching the spin-glass transition from above, we use the small-p expansion of $I(p)$ in $D = 4$,

$$I(p) \simeq C\left(\ln\left(\frac{1}{\lambda}\right) - 1\right) - \frac{C_2}{\lambda}p^2 \tag{4.20}$$

where $C_2 > 0$. After defining $\lambda = \lambda_o + \delta\lambda$, $\delta\lambda \to 0^+$, this gives

$$\chi_{22}(p) \simeq \left(\frac{\lambda_o}{8u^2C_2}\right)\frac{1}{p^2 + \frac{C}{C_2}\delta\lambda}. \tag{4.21}$$

Using equations (3.4), (3.10) and (3.12) one finds that when $\tau \to \tau_{sg}^+$

$$\delta\lambda \simeq \frac{2u}{3g}(\tau - \tau_{sg}) \tag{4.22}$$

which finally leads to

$$\chi_{22}(p) \simeq \left(\frac{\lambda_o}{8u^2C_2}\right)\frac{1}{p^2 + \frac{2uC}{3gC_2}(\tau - \tau_{sg})}. \tag{4.23}$$

Consider now the spin-glass phase (in $D = 4$). One then has $2uI(0) = 1$, $\lambda = \lambda_o$, and $\mu = \frac{2u}{3g}(\tau_{sg} - \tau)$, so that for small p,

$$I(p) \simeq \frac{1}{2u} - \frac{C_2}{\lambda_o}p^2 \tag{4.24}$$

$$J(p) \simeq \frac{C}{2\lambda_o} - \frac{C_2}{2\lambda_o}p^2. \tag{4.25}$$

The determinant of the matrix $\mathbf{M}(p)$, equation (4.16), can now be expressed in the leading order as

$$\det(\mathbf{M}(p)) \simeq \frac{3gC_2}{\lambda_o}\left(p^2 + \frac{2uC}{3gC_2}(\tau_{sg} - \tau)\right) \tag{4.26}$$

so that when $\tau \to \tau_{sg}^-$, the susceptibilities $\chi_{11}(p)$ stay, $\chi_{12}(p)$ are finite, whereas

$$\chi_{22}(p) \simeq \left(\frac{\lambda_o}{8u^2C_2}\right)\frac{1}{p^2 + \frac{2uC}{3gC_2}(\tau_{sg} - \tau)}. \tag{4.27}$$

One concludes from the above formulae that the susceptibility associated with the external field Δ that couples to the Edwards–Anderson order parameter q, i.e. $\chi_{22}(p) = \frac{\delta^2 F}{\delta\Delta(-p)\delta\Delta(p)}$, diverges when the critical point τ_{sg} is approached both from above and below as $|\tau - \tau_{sg}|^{-1}$ (in $D = 4$), whereas the associated correlation length (that characterizes the long-distance behaviour of the two-point composite-field correlation function, equation (4.13)) diverges as $|\tau - \tau_{sg}|^{-1/2}$ (in $D = 4$). The corresponding critical exponents, $\gamma = 1$, $\nu = 1/2$, are thus classical.

5. Discussion

5.1. Dimensions other than $D = 4$

The situation in dimensions $D > 4$ is quite simple. There, the value of the integral

$$\left[G_c^2\right] \equiv \int_{|p|<1} \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{p^2 + \lambda} \tag{5.1}$$

remains finite (not diverging) in the limit $\lambda \to 0$. Therefore, when $u \ll 1$ the only solution of the saddle-point equation (3.7) for the spin-glass order parameter is trivial, $\mu = 0$, so that there is no spin-glass solution in dimensions $D > 4$. Thus, one recovers in this case the standard scenario: upon lowering the temperature, the only phase transition that takes place in the system is a second-order phase transition from the paramagnetic to the ferromagnetic phase, and this phase transition is described by the Gaussian theory.

On the other hand, the phase diagram in dimensions $D < 4$ turns out to be similar to that in $D = 4$ at a qualitative level. The integral, equation (5.1), diverges in the limit $\lambda \to 0$, and, therefore, when $u \ll 1$ there is always a spin-glass solution $\mu \neq 0$ of the saddle-point equation (3.7). Thus, in this case one recovers *within the Gaussian variational approximation* a phase diagram similar to that in dimension $D = 4$, where the paramagnetic phase is separated from the ferromagnetic one by an intermediate spin-glass phase.

5.2. Validity of the Gaussian variational approximation

Since the main result of the present study, namely, the existence of a spin-glass phase separating the paramagnetic and ferromagnetic phases, is in apparent contradiction with the generally accepted view on the phase diagram of the disordered Ising ferromagnet, the limits of validity of the Gaussian variational approximation used in this paper require a detailed study.

It is well known that the Gaussian variational approach becomes exact when the number of spin components tends to infinity (see footnote 2). Otherwise (in particular, the Ising model is very far from this limit) it is not more than an approximation characterized by certain bounds of validity (if any), and ignoring these bounds may just lead to wrong conclusions. The typical example is the pure Ising model in dimensions $D \leqslant 4$: there, one can easily check (using equations (2.31) and (2.33) with $u = \mu = 0$) that according to the Gaussian variational approximation the transition from the paramagnetic to the ferromagnetic phase turns out to be first order, which is of course incorrect.

The validity of the first-order cumulant approximation in the deviation $\left(H^{(n)} - H_g^{(n)}\right)$, described in section 2, can be checked by estimating the contribution from the higher order terms. In the vicinity of the critical temperature T_c of the (supposed) paramagnetic–ferromagnetic phase transition, for $|T - T_c|/T_c \equiv \tau \ll 1$, a systematic account of these contributions can be done in terms of the usual renormalization group (RG) procedure, which yields an effective scale dependence of the renormalized non-Gaussian interaction parameters g and u. In dimensions $D = 4$, in the so-called one-loop approximation, the scale evolution of g and u is described by the following well-known RG equations (see e.g. [1, 6, 12]):

$$\frac{\mathrm{d}}{\mathrm{d}\xi} g(\xi) = -6C(3g - 4u)g + O(g^3; g^2u; gu^2; u^3) \tag{5.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi} u(\xi) = -4C(3g - 4u)u + O(g^3; g^2u; gu^2; u^3) \tag{5.3}$$

where, as usual, $C = 1/16\pi^2$, and $\xi \equiv \ln L$ is the standard RG rescaling parameter which is equal to the logarithm of the spatial scale. According to these equations one can conclude

that the renormalization of the parameters g and u remains irrelevant (so that the higher order terms of the perturbation theory are not important) and that the theory remains Gaussian at scales bounded by the conditions

$$6C|3g_0 - 4u_0|g_0\xi \ll g_0 \tag{5.4}$$

$$4C|3g_0 - 4u_0|u_0\xi \ll u_0 \tag{5.5}$$

where $g_0 \equiv g(\xi = 0)$ and $u_0 \equiv u(\xi = 0)$ are the 'bare' (microscopic) values of the parameters g and u. These two conditions are satisfied when

$$\xi \ll \xi_* \sim \frac{1}{6C|3g_0 - 4u_0|} \tag{5.6}$$

or, in terms of the spatial scale L, until

$$L \ll L_* \sim \exp\left\{\frac{1}{6C|3g_0 - 4u_0|}\right\}. \tag{5.7}$$

Since the temperature and the spatial scales are related by $|\tau| \sim L^{-2}$, one finds that the higher order non-Gaussian corrections remain irrelevant only at temperatures not too close to the (supposed) paramagnetic–ferromagnetic critical point:

$$|\tau| \gg \tau_* \sim \exp\left\{-\frac{1}{3C|3g_0 - 4u_0|}\right\}. \tag{5.8}$$

Note that in the special case $u_0 = \frac{3}{4}g_0$, the higher order terms of the RG equations (5.2), (5.3) come into play, and instead of the condition (5.8) one eventually obtains: $\tau_* \sim \exp\{-(\text{const})/g_0^2\}$. Note also that in the case of the pure system, $u_0 \equiv 0$, equation (5.8) yields the well-known temperature scale $\tau_* \sim \exp\{-16\pi^2/9g_0\}$, such that when $|\tau| \gg \tau_*$ the behaviour of the system is effectively Gaussian, while in the close vicinity of the critical point, for $|\tau| \ll \tau_*$, the non-Gaussian fluctuations become dominant; in particular, this shows that the first-order Gaussian cumulant approximation breaks down for $|\tau| \ll \tau_*$ (it is at this temperature scale that this approximation wrongly predicts the first-order nature of the phase transition).

According to the calculations of section 3, the spin-glass phase separating the paramagnetic and ferromagnetic phases exists in a temperature interval of the order of $\lambda_o \simeq \exp\{-\frac{1}{2uC}\}$ around T_c (where T_c is the critical temperature of the putative paramagnetic–ferromagnetic phase transition). Since the approximation used in the present approach is valid only outside the temperature given by equation (5.8), we need this 'dangerous' temperature interval to be well inside the spin-glass phase, i.e.,

$$\exp\left\{-\frac{1}{3C|3g_0 - 4u_0|}\right\} \ll \exp\left\{-\frac{1}{2uC}\right\}. \tag{5.9}$$

If the above condition is satisfied, the spin-glass state (obtained in terms of the present Gaussian variational approximation) would set in when lowering the temperature well before the non-Gaussian critical fluctuations of the fields $\phi_a(x)$ (given by the higher order terms of the perturbation theory) become relevant.

It should be stressed, however, that this does not guarantee that the critical behaviour at the spin-glass phase transition itself is correctly described by the Gaussian theory (as it is derived in section 4), since for an adequate description of this phase transition one needs to take into account critical fluctuations of the spin-glass order parameter $q_{ab}(x) = \phi_a(x)\phi_b(x)$ and not just those of the fields $\phi_a(x)$. Actually, although less likely, one cannot exclude that the fluctuations of the composite field $q_{ab}(x)$ will even destroy the spin-glass phase itself. This question is left for future investigations.

In addition, when lowering the temperature further, the transition point from the spin-glass to the ferromagnetic phase is also separated from T_c by an interval of the order of λ_o.

Therefore, provided the condition (5.9) is satisfied, one can conclude that the first-order nature of this phase transition corresponds indeed to the proper physical phenomenon, and is not just an artefact of the method used.

Since all the calculations of the present study are done for $g \ll 1$ and $u \ll 1$, one can easily see that the condition (5.9) is satisfied provided

$$\frac{10}{9} < \frac{g_0}{u_0} < \frac{14}{9} \tag{5.10}$$

so that the parameter u describing the disorder strength must be of the same order as the interaction parameter g of the pure system. It is in this sense that we characterize such system as having a *finite* strength of disorder (although both u and g are kept small).

Thus, we have found at least a limited region of the parameters, defined by equation (5.10), where the approximation used in this paper appears to be reasonable and for which an intermediate spin-glass phase could exist. Presumably, it could also exist in a wider region (in particular for large values of u compared to g), where, however, it cannot be studied by the present method. Unfortunately, in the framework of the present investigation we cannot answer the question whether the (exponentially narrow) intermediate spin-glass phase continues to exist in the limit $u \to 0$ or there exists a critical value u_c such that the spin-glass phase appears only for $u > u_c$.

At a qualitative level, the situation in dimensions $D = 4 - \epsilon (\epsilon \ll 1)$ turns out to be similar to that in $D = 4$ (see appendix C): we find again the same restriction on the parameters g and u as in equation (5.10), but in addition we obtain that the value of the parameter $\epsilon = 4 - D$ must remain small. This shows that on the one hand the results obtained in the present study are not unique to the dimension $D = 4$ and can be continued down to lower dimensions; on the other hand, due to the restriction $\epsilon \ll 1$, we cannot guarantee that they would survive down to the physical dimension $D = 3$ (where, for sure, the present approximation does not work).

5.3. Conclusion

The existence of a spin-glass phase in a 'dilute ferromagnet' is *a priori* a puzzle [4, 8]. In the absence of competing antiferromagnetic interactions and of the associated frustration, how can the system ever be found in a state with frozen magnetizations but without global magnetization?[3] Indeed, for a spin-glass phase as predicted by the Gaussian variational approximation to exist in a disordered Ising model, one must have

(i) non-zero 'frozen' local magnetizations in an extensive part of the volume, so that the Edwards–Anderson order parameter is non-zero;
(ii) a total magnetization equals to zero.

Truly frozen local magnetizations require symmetry breaking and can exist only on a percolating cluster that diverges in the thermodynamic limit. If L is the linear size of the whole system (ultimately, $L \to \infty$), the size of clusters with non-zero magnetization must scale as L^d with $D \geqslant d > 1$ (for Ising spins). On the other hand, on each cluster the magnetization can be either positive or negative. One possible scenario to explain the occurrence of the spin-glass phase with zero total magnetization is thus as follows.

We envisage that there exists a global temperature at which a large number M of percolating clusters of fractal dimension $d < D$ acquire a non-zero magnetization (when $L \to \infty$), while being essentially decoupled from each other. These clusters are formed from

[3] The fact that at and around the spin-glass transition, the 4-spin spin-glass susceptibility has to be larger than the square of the 2-spin ferromagnetic susceptibility may not be as severe a problem: the presence of a large number of 'ferromagnetic islands' and clusters invalidates the inequalities obtained via a perturbative analysis [8].

the 'islands' characterized by a predominantly negative value of the local temperature $(\tau - \delta\tau)$ and are separated from each other by regions of high local temperature. In zero external magnetic field[4] the sign of the magnetization on each cluster is then randomly determined (and subsequently frozen when the temperature is decreased). Since M is large, there will be essentially (up to a relative factor of the order $1/\sqrt{M}$) as many positive as negative clusters. Therefore, if M scales with L as $M \sim L^{D-d}$ (and $d < D$), the total magnetization, averaged over all clusters, is zero whereas the Edwards–Anderson order parameter is non-zero: the system is then in a spin-glass phase. It must be stressed that the appearance of the large number of percolating clusters cannot be envisaged as a purely geometric, disorder-controlled phenomenon. Thermal fluctuations come into play for helping the islands to coalesce. This is necessary in order to have M scale as L^{D-d} since in usual percolation the number of incipient spanning clusters at the percolation threshold can be larger but in $D < 6$ does not diverge with L as a power law [13].

To settle the question of the existence of a spin-glass phase in the random temperature Ising ferromagnet, especially in $D = 3$, further investigation is clearly needed. A potential line of investigation is a renormalization group analysis that treats on equal footing the primary fields $\phi_a(x)$ and the composite fields $\phi_a(x)\phi_b(x)$ that are needed to describe the spin-glass phase. This would clarify the problem of the upper critical dimension of the paramagnetic to spin-glass phase transition (recall that in our study $D = 4$ is the critical dimension above which the spin-glass phase ceases to exist, it is not necessarily the upper critical dimension). An interesting problem that is worth pursuing is the connection to the phase behaviour of the random field Ising model. It has been suggested [14, 15] that this latter system possesses an intermediate spin-glass-like phase, and that a potential source for the presence of this phase is the generation in higher orders of perturbation theory of 'attractive' terms of the type $u\phi_a^2\phi_b^2$ [15], i.e., terms similar to those present in the random temperature replica Hamiltonian. Although the kind of symmetry breaking leading to the spin-glass phase in the random temperature Ising model is not possible in the random field models (within the replica formulation, random fields generate a nonzero field Δ that is linearly coupled to the composite operator $\phi_a(x)\phi_b(x)$), a comparative study may prove fruitful. Finally, one should stress that absence of the replica symmetry breaking found within the present approach could turn out to be an artefact of the Gaussian variational approximation because the corresponding saddle-point equations (see appendix A) are in fact rather 'fragile' with respect to the introduction of higher order terms of the perturbation theory.

Acknowledgments

The authors are grateful to E Brézin and C De Dominicis for stimulating discussions.

Appendix A

Consider again the saddle-point equations (2.26) in the spin-glass state ($m = 0$):

$$\mu_{ab} = g[\tilde{G}]\delta_{ab} + 2g_{ab}[G_{ab}] \tag{A.1}$$

where

$$G_{ab}^{-1}(p) = (p^2 + \tau)\delta_{ab} + \mu_{ab} \tag{A.2}$$

and $\tilde{G}(p) \equiv G_{a=b}(p)$.

[4] If one imposes a small external magnetic field, all cluster magnetizations will have the same sign, i.e. the sign of the external field. Then, in the limit of vanishingly small field, the overall spontaneous magnetization does not vanish. The situation, however, is different if one considers from the beginning the case without any applied magnetic field.

A.1. Continuous RSB

In the case of a Parisi-like continuous RSB in the limit $n \to 0$ the matrices G_{ab} and μ_{ab} are parametrized by their diagonal elements $\tilde{G}$, $\tilde{\mu}$ and by off-diagonal *functions* $G(p; x)$, $\mu(p; x)$ defined on the interval $x \in [0, 1]$ [11]:

$$G_{ab}(p) \to (\tilde{G}(p); G(p; x)) \tag{A.3}$$

$$\mu_{ab}(p) \to (\tilde{\mu}(p); \mu(p; x)). \tag{A.4}$$

By using the Parisi algebra for inverting hierarchical matrices [10] one derives from equation (A.2)

$$G(p; x) = -\frac{\mu(0)}{(p^2 + \tau + \tilde{\mu} - \bar{\mu})^2} - \int_0^x \mathrm{d}y \frac{\mu'(y)}{(p^2 + \lambda(y))^2} \tag{A.5}$$

where

$$\lambda(y) \equiv \tau + \tilde{\mu} - \bar{\mu} - y\mu(y) + \int_0^y \mathrm{d}z\, \mu(z) \tag{A.6}$$

$$\bar{\mu} \equiv \int_0^1 \mathrm{d}x\, \mu(x) \tag{A.7}$$

$$\mu(0) \equiv \mu(x = 0). \tag{A.8}$$

Substituting equation (A.5) into equation (A.1) (for $a \neq b$), we find the following equation for the unknown function $\mu(x)$:

$$\mu(x) = 2u \int_{|p|<1} \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{\mu(0)}{(p^2 + \tau + \tilde{\mu} - \bar{\mu})^2} + 2u \int_{|p|<1} \frac{\mathrm{d}^D p}{(2\pi)^D} \int_0^x \mathrm{d}y \frac{\mu'(y)}{(p^2 + \lambda(y))^2}. \tag{A.9}$$

Differentiating both sides of this equation with respect to x then leads to

$$\mu'(x) \left[2u \int_{|p|<1} \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{(p^2 + \lambda(x))^2} - 1 \right] = 0. \tag{A.10}$$

Thus, either one has $\mu'(x) = 0$, i.e., $\mu(x) = \text{const}$ (independent of x), or

$$2u \int_{|p|<1} \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{(p^2 + \lambda(x))^2} - 1 = 0 \tag{A.11}$$

which also yields $\mu(x) = \text{const}$. In both cases the solution is replica-symmetric (or step-like). This proves that the saddle-point equation (A.1) cannot have solutions with continuous RSB.

Nevertheless, the above proof does not exclude the possibility of other types of solutions with step-like RSB.

A.2. One-step RSB

We consider now a one-step RSB ansatz for the replica matrix $G_{ab}(p)$: it is defined in terms of one diagonal element $\tilde{g}(p)$, *two* off-diagonal elements $g_1(p)$ and $g_0(p)$ and the coordinate of the step x_o:

$$G_{ab}(p) = \begin{cases} \tilde{g}(p) & \text{for } a = b \\ g_1(p) & \text{for } I\left(\dfrac{a}{x_o}\right) = I\left(\dfrac{b}{x_o}\right) \\ g_0(p) & \text{for } I\left(\dfrac{a}{x_o}\right) \neq I\left(\dfrac{b}{x_o}\right) \end{cases} \tag{A.12}$$

where $I(x)$ is the integer valued function which is equal to the smallest integer larger than or equal to x. Substituting the above equation into the general expression, equation (2.19), with

$m = 0$, one finds the following explicit expression for the free energy density:

$$F[\tilde{g}(p); g_1(p); g_0(p); x_o] = -\frac{1}{2x_o}[\ln(\tilde{g} - g_1 + x_o(g_1 - g_0))] - \frac{1}{2}\left(1 - \frac{1}{x_o}\right)[\ln(\tilde{g} - g_1)] - \left[\frac{g_0}{2(\tilde{g} - g_1 + x_o(g_1 - g_0))}\right] + \frac{1}{2}[(p^2 + \tau)\tilde{g}] + \frac{1}{4}(3g - 2u)[\tilde{g}]^2 + \frac{1}{2}u(1 - x_o)[g_1]^2 + \frac{1}{2}ux_o[g_0]^2 \tag{A.13}$$

where, as usual, the symbol $[(\ldots)]$ denotes the integration over p, see equation (2.18). Variation of this free energy density with respect to the trial functions $g_0(p)$, $g_1(p)$ and $\tilde{g}(p)$ yields three saddle-point equations

$$\frac{g_0(p)}{(\tilde{g}(p) - g_1(p) + x_o(g_1(p) - g_0(p)))^2} = 2u[g_0] \tag{A.14}$$

$$\frac{g_1(p) - g_0(p)}{(\tilde{g}(p) - g_1(p))(\tilde{g}(p) - g_1(p) + x_o(g_1(p) - g_0(p)))} = 2u[(g_1 - g_0)] \tag{A.15}$$

$$\frac{1}{\tilde{g}(p) - g_1(p)} = p^2 + \tau + (3g - 2u)[\tilde{g}] + 2u[g_1]. \tag{A.16}$$

The last equation can be rewritten as follows:

$$\tilde{g}(p) - g_1(p) = \frac{1}{p^2 + \lambda_1} \equiv G_c(p; \lambda_1) \tag{A.17}$$

where the unknown parameter λ_1 is defined by

$$\lambda_1 = \tau + (3g - 2u)[G_c] + 3g[g_1]. \tag{A.18}$$

From equations (A.15) and (A.14) one finds

$$g_1(p) - g_0(p) = q_1 G_c(p; \lambda_1) G_c(p; (\lambda_1 - q_1 x_o)) \tag{A.19}$$

$$g_0(p) = q_0 G_c^2(p; (\lambda_1 - q_1 x_o)) \tag{A.20}$$

and the unknown parameters q_1 and q_0 are defined by

$$[G_c(\lambda_1) G_c(\lambda_1 - q_1 x_o)] = \frac{1}{2u} \tag{A.21}$$

$$q_0\left(2u\left[G_c^2(\lambda_1 - q_1 x_o)\right] - 1\right) = 0. \tag{A.22}$$

One more saddle-point equation is obtained by taking the derivative of the free energy, equation (A.13), with respect to the parameter x_o; it reads

$$[\ln G_c(\lambda_1)] - [\ln G_c(\lambda_1 - q_1 x_o)] + q_1 x_o [G_c(\lambda_1 - q_1 x_o)] = \frac{1}{4u}(q_1 x_o)^2. \tag{A.23}$$

In this way, we have obtained four equations, (A.18), (A.21)–(A.23), for four unknown parameters, λ_1, q_1, q_0 and x_o.

Introducing the notation $\lambda_1 - q_1 x_o \equiv \lambda_2$ and taking into account the definition of the Green function G_c, equation (A.17), equations (A.21) and (A.23) can be represented as follows:

$$[G_c(\lambda_2)] - [G_c(\lambda_1)] - \frac{1}{2u}(\lambda_1 - \lambda_2) = 0 \tag{A.24}$$

$$\int_{\lambda_2}^{\lambda_1} \mathrm{d}\lambda \left([G_c(\lambda_2)] - [G_c(\lambda)] - \frac{1}{2u}(\lambda - \lambda_2)\right) = 0. \tag{A.25}$$

One can easily check that the function $\psi(\lambda) = [G_c(\lambda_2)] - [G_c(\lambda)] - \frac{1}{2u}(\lambda - \lambda_2)$ has a unique extremum at $\lambda = \lambda_o$ $\left(\text{defined by } \left[G_c^2(\lambda_o)\right] = \frac{1}{2u}\right)$. Therefore, the two equations (A.24) and

(A.25) can be simultaneously satisfied only when $\lambda_1 = \lambda_2$, which means that $q_1 x_o = 0$. In either of the two cases, $q_1 = 0$ or $x_o = 0$, we come back to the replica-symmetric structure of the trial replica matrix G_{ab}.

Quite similar (although more cumbersome) calculations show that there are also no solutions with more than one step of RSB. Thus, within the present approximation we cannot have solutions with broken replica symmetry in the spin-glass phase.

Appendix B

The saddle-point solutions for non-ferromagnetic states ($m = 0$) are defined by two order parameters λ and μ. The stability of these states is determined by the signs of the eigenvalues of the corresponding Hessian:

$$\begin{pmatrix} \frac{\partial^2 f}{\partial \mu^2} & \frac{\partial^2 f}{\partial \mu \partial \lambda} \\ \frac{\partial^2 f}{\partial \lambda \partial \mu} & \frac{\partial^2 f}{\partial \lambda^2} \end{pmatrix}. \tag{B.1}$$

As usual in the replica theory, one should take into account that in the process of taking the limit $n \to 0$ the minima of the replica free energy (at $n > 1$) can turn into maxima (at $n < 1$). The physically relevant states correspond of course to minima of the replica free energy before taking the limit $n \to 0$. Around these states all the eigenvalues of the corresponding Hessian (composed of the second derivatives of the free energy with respect to *all* replica components of the order parameters) must be positive. However, if we consider the expression for the free energy where the limit $n \to 0$ is already taken, the situation can change. In particular, within the replica-symmetric ansatz for the matrix μ_{ab}, equation (2.27), the total number of off-diagonal elements (all equal to $-\mu$) is equal to $n(n-1)$, and this number becomes *negative* for $n < 1$. Therefore, the physically relevant state defined by the free energy in equation (2.34) (where the limit $n \to 0$ is already taken) must be the *maximum* with respect to the parameter μ. On the other hand, the total number of diagonal elements of the matrix μ_{ab} is equal to n, and it remains positive in the limit $n \to 0$. Therefore, the physically relevant extremum of the free energy, equation (2.34), must be the *minimum* with respect to the parameter λ. Thus, in terms of the Hessian, equation (B.1), a physically relevant saddle-point solution must correspond to negative (corresponding to the parameter μ) and one positive (corresponding to the parameter λ) eigenvalue.

By taking the derivatives of the free energy, equation (2.34), with respect to μ and λ and using the saddle-point equations (3.6) and (3.7)), we find (for $m = 0$)

$$\frac{\partial^2 f}{\partial \mu^2} = \frac{3}{2} g [G^2]^2 \tag{B.2}$$

$$\frac{\partial^2 f}{\partial \mu \partial \lambda} = -\frac{1}{2}[G^2](1 + (3g - 2u)[G^2] + 6g\mu[G^3]) \tag{B.3}$$

$$\frac{\partial^2 f}{\partial \lambda^2} = \frac{1}{2}[G^2](1 + (3g - 2u)[G^2]) + \mu[G^3](6g[G^2] - 1 + 6g\mu[G^3]). \tag{B.4}$$

For the paramagnetic state ($\mu = 0$) we obtain

$$\frac{\partial^2 f}{\partial \mu^2} = \frac{3}{2} g [G^2]^2 \tag{B.5}$$

$$\frac{\partial^2 f}{\partial \mu \partial \lambda} = -\frac{1}{2}[G^2](1 + (3g - 2u)[G^2]) \tag{B.6}$$

$$\frac{\partial^2 f}{\partial \lambda^2} = \frac{1}{2}[G^2](1 + (3g - 2u)[G^2]). \tag{B.7}$$

The two eigenvalues of the Hessian, equation (B.1), are then

$$\Lambda_{1,2}^{(P)} = \frac{1}{4}[G^2](1+(6g-2u)[G^2])\left[1 \pm \sqrt{1+\frac{4(1+(3g-2u)[G^2])}{(1+(6g-2u)[G^2])^2}(2u[G^2]-1)}\right]. \quad \text{(B.8)}$$

One of these eigenvalues is always positive, while the other one is negative when $2u[G^2] > 1$; in this case the paramagnetic solution is *stable*. On the other hand, when $2u[G^2] < 1$ the second eigenvalue is also positive, and the paramagnetic solution is *unstable*. Using equation (3.2) we find that the condition $2u[G^2] > 1$ is equivalent to $\lambda(\tau) < \lambda_o$ (equation (3.10)), which is satisfied when $\tau > \tau_{sg}$. Thus, we can conclude that the paramagnetic solution is stable only when $\tau > \tau_{sg}$, while for $\tau < \tau_{sg}$ it becomes *unstable*.

We study now the stability of the spin-glass solution. Substituting equation (3.8) and $[G^3] = C/2\lambda_o$ into equations (B.2)–(B.4) leads to

$$\frac{\partial^2 f}{\partial \mu^2} = \frac{3g}{8u^2} \quad \text{(B.9)}$$

$$\frac{\partial^2 f}{\partial \mu \partial \lambda} = -\frac{3g}{8u^2}\left(1+\frac{2Cu}{\lambda_o}\mu\right) \quad \text{(B.10)}$$

$$\frac{\partial^2 f}{\partial \lambda^2} = \frac{3g}{8u^2}\left(1+\frac{2Cu}{\lambda_o}\mu\right)^2 - \frac{C}{2\lambda_o}\mu. \quad \text{(B.11)}$$

The two corresponding eigenvalues are then

$$\Lambda_{1,2}^{(SG)} = \frac{1}{2}(f''_{\mu\mu}+f''_{\lambda\lambda})\left[1 \pm \sqrt{1+\frac{3Cg}{4u^2\lambda_o(f''_{\mu\mu}+f''_{\lambda\lambda})^2}\mu}\right] \quad \text{(B.12)}$$

where $f''_{\lambda\lambda}$ and $f''_{\lambda\lambda}$ are short-hand notation for $\partial^2 f/\partial\mu^2$ and $\partial^2 f/\partial\lambda^2$, respectively. From the above expression, one can conclude that in the temperature region $\tau < \tau_{sg}$ where the spin-glass solution is physical ($\mu(\tau) > 0$, equation (3.11)), there is always one positive and one negative eigenvalue, which means that the spin-glass solution is *stable*.

Thus for $\tau > \tau_{sg}$ the only stable state of the system is paramagnetic, while for $\tau < \tau_{sg}$ the paramagnetic solution becomes unstable and the stable state of the system is the spin-glass phase.

Appendix C

For the spin-glass state in dimensions $D = 4-\epsilon$ ($\epsilon \ll 1$) one again obtains the solutions, equations (3.8) and (3.9), where instead of equations (3.1) and (3.2) one now has

$$[G_c] = \int_{|p|<1} \frac{\mathrm{d}^D p}{(2\pi)^D}\frac{1}{p^2+\lambda} \simeq C\left(1+\frac{2}{\epsilon}\lambda\left(1-\lambda^{-\epsilon/2}\right)\right)\Big|_{\lambda\ll 1} \simeq C \quad \text{(C.1)}$$

$$\left[G_c^2\right] = \int_{|p|<1} \frac{\mathrm{d}^D p}{(2\pi)^D}\frac{1}{(p^2+\lambda)^2} \simeq C\left(\frac{2}{\epsilon}\left(\lambda^{-\epsilon/2}-1\right)-\lambda^{-\epsilon/2}\right). \quad \text{(C.2)}$$

Then, instead of equation (3.10), equation (3.8) leads to

$$\lambda_o = \left[\frac{4Cu\left(1-\frac{\epsilon}{2}\right)}{\epsilon+4Cu}\right]^{2/\epsilon}. \quad \text{(C.3)}$$

In the limit $\epsilon \ll u \ll 1$ this result reduces back to equation (3.10), so that one eventually recovers all the solutions corresponding to the case $D = 4$ described in section 3.

Let us consider the other limit

$$u \ll \epsilon \ll 1. \tag{C.4}$$

In this case equation (C.3) yields

$$\lambda_o \simeq \left(\frac{4C}{\epsilon}u\right)^{2/\epsilon}. \tag{C.5}$$

Thus, one can easily observe that when $u \ll \epsilon \ll 1$ one recovers all the solutions described in section 3, in which the value of λ_o is now given by equation (C.5). In particular, it is this value of λ_o that controls the temperature range of the spin-glass phase separating the paramagnetic and the ferromagnetic phases.

As in section 5.1 where we discussed the validity of the Gaussian variational approach in dimensions $D = 4$, one should study in place of equations (5.2) and (5.3), the following RG equations:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}g(\xi) = \epsilon g - 6C(3g - 4u)g + O(g^3; g^2u; gu^2; u^3) \tag{C.6}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}u(\xi) = \epsilon u - 4C(3g - 4u)u + O(g^3; g^2u; gu^2; u^3). \tag{C.7}$$

These equations show that the higher order corrections of the perturbation theory remain irrelevant at spatial scales R bounded by the condition

$$6C|3g_0 - 4u_0|R^\epsilon \ll \epsilon \tag{C.8}$$

which corresponds to the temperature scale

$$\tau \gg \tau_* \sim \left(\frac{6C|3g_0 - 4u_0|}{\epsilon}\right)^{2/\epsilon}. \tag{C.9}$$

Thus, as done in section 5.1, to guarantee that the higher order corrections do not destroy the approximation made in the present approach, one needs $\lambda_o \gg \tau_*$, or

$$\left(\frac{4C}{\epsilon}u\right)^{2/\epsilon} \gg \left(\frac{6C|3g_0 - 4u_0|}{\epsilon}\right)^{2/\epsilon}. \tag{C.10}$$

One can easily verify that this condition is satisfied when

$$\frac{10}{9} < \frac{g_0}{u_0} < \frac{14}{9} \tag{C.11}$$

and $\epsilon \ll 1$. In other words, we find the same restriction on the parameters g and u as in $D = 4$, but in addition one needs the value of $\epsilon = 4 - D$ to be small.

References

[1] Harris A B 1974 *J. Phys. C: Solid State Phys.* **7** 1671
Harris A B and Lubensky T C 1974 *Phys. Rev. Lett.* **33** 1540
Khmelnitskii D E 1975 *Sov. Phys.–JETP* **41** 981
Grinstein G and Luther A 1976 *Phys. Rev.* B **13** 1329
Folk R, Holovatch Yu and Yavors'kii T 2001 *Preprint* cond-mat/0106468

[2] Selke W *et al* 1994 *Annual Reviews of Computational Physics* ***I*** ed D Stauffer (Singapore: World Scientific)
Reis F D A, de Queiroz S L A and dos Santos R R 1997 *Phys. Rev.* B **56** 6013
Ballesteros H G, Fernandez L A, Martin-Mayor V, Munoz Sudupe A, Parisi G and Ruiz-Lorenzo J J 1998 *Phys. Rev.* B **58** 2740
Ballesteros H G, Fernandez L A, Martin-Mayor V, Munoz Sudupe A, Parisi G and Ruiz-Lorenzo J J 1997 *J. Phys. A: Math. Gen.* **30** 8379

[3] Marro J, Labarta A and Tejada J 1986 *Phys. Rev.* B **34** 347
Chowdhury D and Stauffer D 1986 *J. Stat. Phys.* **44** 203
Braun P and Fahnle M 1988 *J. Stat. Phys.* **52** 775
Holey T and Fahnle M 1990 *Phys. Rev.* B **41** 11709
[4] Ma S K and Rudnick J 1978 *Phys. Rev. Lett.* **40** 589
[5] Dotsenko Vik S 1999 *J. Phys. A: Math. Gen.* **32** 2949
Bray A J, McCarthy T, Moore M A, Reger J D and Young A P 1987 *Phys. Rev.* B **36** 2212
McKane A J 1994 *Phys. Rev.* B **49** 12003
[6] Dotsenko V S *et al* 1995 *J. Phys. A: Math. Gen.* **28** 3093
Dotsenko V S and Feldman D E 1995 *J. Phys. A: Math. Gen.* **28** 5183
[7] See for instance Korzhenevskii A L, Heuer H-O and Herrmanns K 1998 *J. Phys. A: Math. Gen.* **31** 927
[8] Sherrington D 1980 *Phys. Rev.* B **22** 5553
[9] Edwards S F and Muthukumar M 1988 *J. Chem. Phys.* **89** 2435
Korshunov S 1993 *Phys. Rev.* B **48** 3969
Giamarchi T and Le Doussal P 1995 *Phys. Rev.* B **52** 1242
Dean D S and Lancaster D 1996 *Phys. Rev. Lett.* **77** 3037
[10] Mezard M and Parisi G 1991 *J. Physique.* I **1** 809
[11] Mézard M, Parisi G and Virasoro M A 1987 *Spin Glass Theory and Beyond* (Singapore: World Scientific)
[12] Dotsenko V S 2001 *Introduction to the Replica Theory of Disordered Statistical Systems* (Cambridge: Cambridge University Press)
[13] Aizenman M 1996 *Preprint* cond-mat/9609240
[14] Yoshizawa H and Belanger D 1984 *Phys. Rev.* B **30** 5220
Shapir Y 1984 *J. Phys. C: Solid State Phys.* **17** L809
Ro C, Grest G, Soukoulist C and Levin K 1985 *Phys. Rev.* B **31** 1682
de Almeida J R L and Bruisma R 1987 *Phys. Rev.* B **35** 7267
Guagnelli M, Marinari E and Parisi G 1993 *J. Phys. A: Math. Gen.* **26** 5675
De Dominicis C, Orland H and Temesvari T 1996 *J. Physique* I **5** 987
[15] Brézin E and De Dominicis C 2000 Interactions of several replicas in the random field Ising model *Preprint* cond-mat/0007457